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THE k^{th} PRIME IS GREATER THAN $k(\ln k + \ln\ln k - 1)$ FOR $k \geq 2$

PIERRE DUSART

ABSTRACT. ROSSER and SCHOENFELD have used the fact that the first 3,500,000 zeros of the RIEMANN zeta function lie on the critical line to give estimates on $\psi(x)$ and $\theta(x)$. With an improvement of the above result by BRENT *et al.*, we are able to improve these estimates and to show that the k^{th} prime is greater than $k(\ln k + \ln\ln k - 1)$ for $k \geq 2$. We give further results without proof.

1. INTRODUCTION

For simplicity of notation, we write $\ln_2 x$ instead of $\ln\ln x$.

The purpose of this paper is to find a lower bound of p_k, defined as the k^{th} prime ($p_1 = 2$). The prime number theorem implies that

$$p_k \sim k \ln k.$$

Moreover, the asymptotic expansion of p_k is well-known. CIPOLLA ([2]) gave the first terms in 1902:

$$p_k = k\left\{\ln k + \ln_2 k - 1 + \frac{\ln_2 k - 2}{\ln k} - \frac{\ln_2^2 k - 6\ln_2 k + 11}{2\ln^2 k} + O\left(\left(\frac{\ln_2 k}{\ln k}\right)^3\right)\right\}.$$

ROSSER ([6]) showed that $p_k \geq k \ln k$ for $k \geq 1$, and made his result more precise with SCHOENFELD ([9]) by showing that $p_k \geq k(\ln k + \ln_2 k - 3/2)$ for $k \geq 2$. In 1983, ROBIN ([5]) succeeded in proving that

$$p_k \geq k(\ln k + \ln_2 k - 1.0072629).$$

MASSIAS and ROBIN ([4]) were able to show that

$$p_k \geq k(\ln k + \ln_2 k - 1)$$

for $2 \leq k \leq \exp(598)$ and $k \geq \exp(1800)$. Now we are able to show this for all $k \geq 2$. The constant "1" is very interesting because it appears in the asymptotic expansion.

The proof is based on an explicit estimate of the CHEBYSHEV function defined by

$$\psi(x) = \sum_{\substack{p,\nu \\ p^\nu \leq x}} \ln p.$$

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Define also

$$\theta(x) = \sum_{p\le x} \ln p.$$

Estimates of $\psi(x)$ in [9] are based on the verification of the RIEMANN hypothesis for the first 3,502,500 zeros of $\zeta(s)$ in the critical strip and on an area without zeros of the same type as the area originally found by DE LA VALLÉE POUSSIN. We will use better knowledge about these zeros to obtain finer results on ψ. BRENT *et al.* [1, 3] have shown that the first 1,500,000,001 zeros are on the critical line. We will deduce from this an effective lower estimate for p_k and give further results without proof.

Let $N(T)$ be the number of zeros $\rho = \beta + i\gamma$ of ζ such that $0 < \gamma \le T$. There is an approximation $F(T)$ of $N(T)$ given in [7]:

$$F(T) = \frac{T}{2\pi}\ln\frac{T}{2\pi} - \frac{T}{2\pi} + \frac{7}{8}.$$

Let A be a real constant such that all zeros $\rho = \beta + i\gamma$ of ζ in the critical strip have $\beta = 1/2$ for $0 < \gamma \le A$. The number A, used by ROSSER and SCHOENFELD, is defined as the unique solution of the equation $F(A) = 3,502,500$, which yields $A = 1,894,438.51224\cdots$ and $F(A) = N(A)$. We use a new value of A, defined as the unique solution of the equation $F(A) = 1,500,000,001$. According to [3], this yields $A = 545,439,823.215\cdots$ and $F(A) = N(A)$.

2. ESTIMATE OF $\psi(x)$

For $\delta > 0$ and $b > 1/2$, let

$$R_m(\delta) = \left((1+\delta)^{m+1}+1\right)^m,$$

$$T_1 = \frac{1}{\delta}\left(\frac{2R_m(\delta)}{2+m\delta}\right)^{1/m},$$

$$R(T) = 0.137\ln T + 0.443\ln_2 T + 1.588,$$

$$K_\nu(z,x) = \frac{1}{2}\int_x^\infty t^{\nu-1}\exp\{(-1/2)z(t+1/t)\}dt \qquad \text{for } z>0, x\ge 0, \nu \text{ real},$$

$$R = 9.645908801,$$

$$X = \sqrt{b/R},$$

$$\phi_m(y) = y^{-m-1}\exp(-X^2/\ln(y/17)).$$

Theorem 1 (Theorem 4 of [9]). *Let $b > 1/2$, m a positive integer, δ a positive real such that $0 < \delta < (1-e^{-b})/m$ and $T_1 \ge 158.84998$. Let $z = 2\sqrt{mb/R}$, $A' = (2m/z)\ln(A/17)$, $Y = \max\{A, 17\exp\sqrt{b/((m+1)R)}\}$,*

$$\Omega_1 = \frac{2+m\delta}{4\pi}\left\{\left(\ln\frac{T_1}{2\pi}+\frac{1}{m}\right)^2 + 0.038207 + \frac{1}{m^2} - \frac{2.82m}{(m+1)T_1}\right\},$$

$$\Omega_2 = \frac{(0.159155)R_m(\delta)z}{2m^2 17^m}\left\{zK_2(z,A') + 2m\left(\ln\frac{17}{2\pi}\right)K_1(z,A')\right\} + R_m(\delta)\{2R(Y)\phi_m(Y) - R(A)\phi_m(A)\},$$

and

$$\varepsilon = \Omega_1 e^{-b/2} + \Omega_2\delta^{-m} + m\delta/2 + e^{-b}\ln(2\pi).$$

Then

$$|\,\psi(x) - x\,| < \varepsilon x \quad \textit{for all } x \ge e^b.$$

Theorem 2. *For* $x \geq \exp(50)$,

$$| \psi(x) - x | \leq 0.905 \cdot 10^{-7} x.$$

Proof. Apply Theorem 1 with $A = 545,439,823.215\cdots$ as defined in Section 1. Furthermore, choose $b = 50$, $\delta = 0.947265625 \cdot 10^{-8}$ and $m = 18$. Evaluation of the quantities in Theorem 1, for example with the computer algebra system MAPLE, yields a value of ε which is slightly smaller than $0.905 \cdot 10^{-7}$. This value has been verified with the help of the GP/PARI package. □

3. Lower estimate for p_k

We already know the following relations ([6, 7, 8, 4]):

Lemma 1.

$$\begin{array}{llll} p_k & \geq & k(\ln k) & \text{for } k \geq 2, \\ p_k & \leq & k(\ln k + \ln_2 k) & \text{for } k \geq 6, \\ p_k & \leq & k \ln p_k & \text{for } k \geq 4, \\ p_k & \geq & k(\ln p_k - 2) & \text{for } k \geq 5. \end{array}$$

Theorem 3. *For* $k \geq 2$,

$$p_k \geq k(\ln k + \ln_2 k - 1).$$

Proof. First observe that Lemma 3, p. 375, of [5] gives the result for $3 \leq p_k \leq 10^{11}$.

- For $10^{11} \leq p_k \leq \exp(500)$, bounds for $\theta(x)$ in [10], p. 357, give

$$\theta(p_k) - p_k \leq c \frac{p_k}{\ln p_k}, \qquad \text{where } c = 0.0077629.$$

Since $p_k \leq k \ln p_k$ for $k \geq 4$, we obtain

$$p_k \geq \theta(p_k) - ck.$$

According to [5], p. 376,

$$\theta(p_k) \geq k\left(\ln k + \ln_2 k - 1 + \frac{\ln_2 k - 2.1454}{\ln k}\right) \quad \text{for } k \geq 3. \tag{1}$$

We deduce that

$$p_k \geq k\left(\ln k + \ln_2 k - 1 + \frac{\ln_2 k - 2.1454}{\ln k} - c\right).$$

The map $k \longmapsto \frac{\ln_2 k - 2.1454}{\ln k}$ is decreasing for $\ln k > \exp(3.1454)$; in the interval of k-values for which $10^{11} \leq p_k \leq \exp(1800)$, its minimum is greater than c.

- For $p_k \geq \exp(1800)$, we borrow from [10], p. 360,

$$|\theta(p_k) - p_k| \leq \eta_4 \frac{p_k}{\ln^4 p_k} \leq \eta_4 \frac{k}{\ln^3 p_k} \tag{2}$$

with $\eta_4 = 1.657 \cdot 10^7$. Substituting (1) in (2) gives

$$p_k \geq \theta(p_k) - \frac{\eta_4}{1800^2} \frac{k}{\ln k} \geq k\left(\ln k + \ln_2 k - 1 + \frac{\ln_2 k - 7.26}{\ln k}\right).$$

For $p_k \geq \exp(1800)$ we have $\ln_2 k > 7.49$, and the result follows.

- We conclude the proof for $\exp(500) < p_k < \exp(1800)$. Since $\theta(x) < \psi(x)$, it derives from Theorem 2 that

$$\theta(p_k) - p_k < \psi(p_k) - p_k < \varepsilon p_k,$$

with $\varepsilon = 0.905 \cdot 10^{-7}$. Now,

$$p_k \geq k\left(\ln k + \ln_2 k - 1 + \frac{\ln_2 k - 2.1454}{\ln k}\right) - \varepsilon p_k,$$

$$p_k \geq k\left(\ln k + \ln_2 k - 1 + \frac{\ln_2 k - 2.1454}{\ln k} - \varepsilon \frac{p_k}{k}\right).$$

It remains to prove that

$$\frac{\ln_2 k - 2.1454}{\ln k} - \varepsilon \frac{p_k}{k} \geq 0.$$

As $p_k \leq k(\ln k + \ln_2 k)$, we must have for $\ln p_k \leq 1800$

$$\varepsilon \leq \frac{\ln_2 k - 2.1454}{\ln k(\ln k + \ln_2 k)}.$$

This holds because ε is less than $1.6 \cdot 10^{-6}$. □

Remark 1. In the interval $]\exp(500), \exp(1800)[$, the values of ε computed by Rosser and Schoenfeld do not seem to be sufficient.

4. Further Results

We now give new bounds for $\psi(x)$ and $\theta(x)$. By computing other values of ε and applying the same kind of ideas that are used above, one can show the following assertions:

$$|\theta(x) - x| \leq 0.006788 \frac{x}{\ln x} \qquad \text{for } x \geq 2.89 \cdot 10^7,$$

and

$$\begin{array}{lll} p_k \leq k(\ln k + \ln_2 k - 0.9484) & & \text{for } k \geq 39017, \\ p_k \leq k\left(\ln k + \ln_2 k - 1 + \frac{\ln_2 k - 1.8}{\ln k}\right) & & \text{for } k \geq 27076, \\ p_k \geq k\left(\ln k + \ln_2 k - 1 + \frac{\ln_2 k - 2.25}{\ln k}\right) & & \text{for } k \geq 2. \end{array}$$

We can use these results to show that, for $x \geq 3275$, $[x, x + x/(2\ln^2 x)]$ contains at least one prime. To conclude, define, as usual, $\pi(x)$ to be the number of primes lower than x. Then computations similar to the ones in [8] lead to

$$\frac{x}{\ln x}\left(1 + \frac{0.992}{\ln x}\right) \underset{x \geq 599}{\leq} \pi(x) \underset{x>1}{\leq} \frac{x}{\ln x}\left(1 + \frac{1.2762}{\ln x}\right).$$

References

[1] R. P. Brent, J. van de Lune, H. J. J. te Riele & D. T. Winter, *On the Zeros of the Riemann Zeta Function in the Critical Strip. II*, Math. Of Computation **39** Number 160 (October 1982), 681-688. MR **83m:**10067; MR **87e:**11103

[2] M. Cipolla, *La determinazione assintotica dell'n^{imo} numero primo*, Matematiche Napoli **3** (1902), 132-166.

[3] J. van de Lune, H. J. J. te Riele & D.T. Winter, *On the Zeros of the Riemann Zeta Function in the Critical Strip.IV*, Math. Of Computation **46** Number 174 (April 1986), 667-681. MR **87e:**11102

[4] J.-P. Massias & G. Robin, *Bornes effectives pour certaines fonctions concernant les nombres premiers*, Journal de Théorie des Nombres de Bordeaux **8** (1996), 215–242. MR **97g:**11099

[5] G. Robin, *Estimation de la fonction de Tchebychef θ sur le k-ième nombre premier et grandes valeurs de la fonctions ω(n), nombre de diviseurs premiers de n*, Acta Arithmetica **42** numéro 4 (1983), 367-389. MR **85j:**11109

[6] J. B. Rosser, *The n-th prime is greater than n log n*, Proc. London Math. Soc. (2) **45** (1939), 21-44.

[7] J. B. Rosser, *Explicit bounds for some functions of prime numbers*, Amer. J. Math. **63** (1941), 211-232. MR **2:**150e

[8] J. B. Rosser & L. Schoenfeld, *Approximate Formulas for Some Functions of Prime Numbers*, Illinois Journal Math. **6** (1962), 64-94. MR **25:**1139

[9] J. B. Rosser & L. Schoenfeld, *Sharper Bounds for the Chebyshev Functions $\theta(x)$ and $\psi(x)$*, Math. Of Computation **29** Number 129 (January 1975), 243-269. MR **56:**11581a

[10] L. Schoenfeld, *Sharper Bounds for the Chebyshev Functions $\theta(x)$ and $\psi(x)$.II*, Math. Of Computation **30** Number 134 (April 1976), 337-360. MR **56:**11581b,c

LACO, ESA 6090, Faculté des Sciences, 123 avenue Albert Thomas, 87060 Limoges Cedex, FRANCE

E-mail address: dusart@unilim.fr